DeGiorgi-Nash lecture notes

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1 Statement of theorems

Theorem 1. *Suppose $u \in W^{1,2}(B_R(x_0))$ such that $u \geq 0$ in $B_R(x_0)$ and*

$$Lu = D_i(a^{ij} D_j u + b^i u) + c^j D_j u + du \geq D_i f^i + g \text{ weakly in } B_R(x_0),$$

where $a^{ij} \in L^\infty(B_R(x_0))$, $b^i, c^i, f^i \in L^q(B_R(x_0))$, and $d, g \in L^{q/2}(B_R(x_0))$ for $q > n$ such that

$$a^{ij}(x)\xi_i \xi_j \geq \lambda |\xi|^2 \text{ for a.e. } x \text{ and } \xi \in \mathbb{R}^n \tag{1}$$

for some constant $\lambda > 0$ and

$$\sum_{i,j=1}^n |a^{ij}|^2 \leq \Lambda^2, \quad \lambda^{-2} R^{2-2n/q} \sum_{i,j=1}^n (\|b^i\|_{L^q}^2 + \|c^i\|_{L^q}^2) + \lambda^{-1} R^{2-2n/q} \|d\|_{L^{q/2}} \leq \nu \tag{2}$$

on $B_R(x_0)$ for some constants $\Lambda, \nu \in (0, \infty)$. Then $u \in L^\infty(B_{R/2}(x_0))$ such that for $p > 1$,

$$\sup_{B_{R/2}(x_0)} u \leq C(R^{-n/p} \|u\|_{L^p(B_R(x_0))} + \lambda^{-1} R^{1-n/q} \|f\|_{L^q(B_R(x_0))} + \lambda^{-1} R^{2-2n/q} \|g\|_{L^{q/2}(B_R(x_0))})$$

for some constant $C = C(n, q, p, \Lambda/\lambda, \nu) \in (0, \infty)$.

Theorem 2 (Weak Harnack inequality)**.** *Suppose $u \in W^{1,2}(B_{2R}(x_0))$ such that $u \geq 0$ in $B_{2R}(x_0)$ and*

$$Lu = D_i(a^{ij} D_j u + b^i u) + c^j D_j u + du \leq D_i f^i + g \text{ weakly in } B_{2R}(x_0),$$

where $a^{ij} \in L^\infty(B_{2R}(x_0))$, $b^i, c^i, f^i \in L^q(B_{2R}(x_0))$, and $d, g \in L^{q/2}(B_{2R}(x_0))$ for $q > n$ such that (1) and (2) hold (on $B_{2R}(x_0)$) for some constants $\lambda, \Lambda, \nu \in (0, \infty)$. Then for $1 \leq p < n/(n-2)$,

$$R^{-n/p} \|u\|_{L^p(B_R(x_0))} \leq C(\inf_{B_{R/2}(x_0)} u + \lambda^{-1} R^{1-n/q} \|f\|_{L^q(B_R(x_0)} + \lambda^{-1} R^{2-2n/q} \|g\|_{L^{q/2}(B_R(x_0))})$$

for some constant $C = C(n, q, p, \Lambda/\lambda, \nu) \in (0, \infty)$.

Observe that Theorem 1 says remarkably that if we have a subsolution $u \in W^{1,2}$, which needn't be bounded, then in fact u is bounded. In fact we will later use Theorem 2 to say something slightly stronger, namely that any solution $u \in W^{1,2}$ to an elliptic equation is in fact Hölder continuous. In the proof, we will show that u is bounded by showing $u \in L^p$ for all $p \geq 1$ with the average L^p

norm of u uniformly bounded independent of p. Thus the interesting step here is how we go from $u \in L^2$ to $u \in L^p$ for $p \geq 2$. This is called Moser iteration.

Recall that later we will prove a Hölder continuity estimate using the weak Harnack inequality. In the case of quasilinear elliptic equations, which is a class of nonlinear elliptic equations including the minimal surface equation, by differentiating the quasilinear equation one can use this to prove $C^{1,\mu}$ estimates on solutions. This is the key to the Leray-Schauder existence theory for quasilinear elliptic equations. Using the maximum principle and energy estimates, one can establish C^1 bounds on solutions to quasilinear elliptic equations in divergence form, however the Leray-Schauder theory requires a $C^{1,\mu}$ bound.

Before looking at the proof, observe that by combining Theorems 1 and 2, we obtain:

Corollary 1 (Harnack inequality)**.** *Let Ω be a connected open set in $\mathbb{R}^n$. Suppose $u \in W^{1,2}(\Omega)$ such that $u \geq 0$ in Ω and*

$$Lu = D_i(a^{ij} D_j u + b^i u) + c^j D_j u + du = D_i f^i + g \text{ weakly in } \Omega,$$

where $a^{ij} \in L^\infty(\Omega)$, $b^i, c^i, f^i \in L^q(\Omega)$, and $d, g \in L^{q/2}(\Omega)$ for $q > n$ such that (1) holds true on Ω and

$$\sum_{i,j=1}^n |a^{ij}|^2 \leq \Lambda^2, \quad \lambda^{-2} \sum_{i,j=1}^n (\|b^i\|^2_{L^q(\Omega)} + \|c^i\|^2_{L^q(\Omega)}) + \lambda^{-1} \|d\|_{L^{q/2}(\Omega)} \leq \nu$$

for some constants $\lambda, \Lambda, \nu \in (0,\infty)$. Then $u \in L^\infty_{loc}(\Omega)$ and for every $\Omega' \subset\subset \Omega$,

$$\sup_{\Omega'} u \leq C(\inf_{\Omega'} u + \lambda^{-1} \|f\|_{L^q(\Omega)} + \lambda^{-1} \|g\|_{L^{q/2}(\Omega)})$$

for $C = C(n, \Lambda/\lambda, \nu, \Omega', \Omega) \in (0,\infty)$.

Proof. By combining Theorems 1 and 2, choosing $p \in (1, n/(n-2))$, for every ball $B_{4R}(x_0) \subset\subset \Omega$,

$$\sup_{B_R(x_0)} u \leq C(\inf_{B_R(x_0)} u + \lambda^{-1} R^{1-n/q} \|f\|_{L^q(\Omega)} + \lambda^{-1} R^{2-2n/q} \|g\|_{L^{q/2}(\Omega)}) \tag{3}$$

for some constant $C = C(n, q, p, \Lambda/\lambda, \nu) \in [2,\infty)$. Cover Ω' be a finite collection of open balls $B_j = B_{R_j}(x_j)$, $j = 1, 2, \ldots, N$, such that $B_{4R_j}(x_j) \subset\subset \Omega$. Assume $\bigcup_{j=1}^N B_j$ is connected, as otherwise $\bigcup_{j=1}^N B_j$ consists of finitely many connected components, each of which can be connected by a continuous path in Ω since Ω is connected and then we can cover the path by additional balls B_j until $\bigcup_{j=1}^N B_j$ is connected. Let $y, z \in \Omega'$ and $\gamma : [0,1] \to \Omega'$ be a path from $\gamma(0) = y$ to $\gamma(1) = z$. It is clear that since the balls B_j cover the image of γ, we can label the balls $B_1, B_2, \ldots, B_M$, where $M \leq N$, and find times $0 = t_0 < t_1 < t_2 < \cdots < t_{M-1} < t_M = 1$ so that $\gamma(t_j) \in B_j \cap B_{j+1}$ for $1 \leq j < M$. Specifically, inductively define B_j and t_j by choosing B_1 containing y and then for each $j \geq 1$ choosing B_{j+1} to contain $\gamma(\tau_j)$, where $\tau_j = \sup\{t \in (t_{j-1}, 1] : \gamma(t) \in B_j$ and then choosing t_j slightly less than τ_j such that $\gamma(t_j) \in B_j \cap B_{j+1}$. By (3),

$$\sup_{B_j} u \leq C(\inf_{B_j} u + \lambda^{-1} \|f\|_{L^q(\Omega)} + \lambda^{-1} \|g\|_{L^{q/2}(\Omega)}) \tag{4}$$

for all $j = 1, 2, \ldots, M$ for some constant $C \in [1,\infty)$ depending only on n, q, Λ/λ, ν, and the radii R_j. After perturbing y, z, and t_j slightly if necessary to account for the fact that u is only defined up to sets of zero measure, (4) implies that

$$u(\gamma(t_j)) \leq C(u(\gamma(t_{j+1})) + \lambda^{-1} \|f\|_{L^q(\Omega)} + \lambda^{-1} \|g\|_{L^{q/2}(\Omega)})$$

for $j = 0, 1, 2, \ldots, M-1$, from which it follows that

$$u(y) \leq C^N u(z) + \sum_{j=1}^{N} C^j \lambda^{-1}(\|f\|_{L^q(\Omega)} + \|g\|_{L^{q/2}(\Omega)}).$$

Taking the essential supremum over $y \in \Omega'$ and the essential infimum over $z \in \Omega'$,

$$\sup_{\Omega'} u \leq C^N \inf_{\Omega'} u + \sum_{j=1}^{N} C^j \lambda^{-1}(\|f\|_{L^q(\Omega)} + \|g\|_{L^{q/2}(\Omega)}).$$

□

2 Part 1 of proof: Weak equation and Moser iteration

In this section we will prove Theorem 1 and begin the proof of Theorem 2. It will be convenient to do this with the two theorems jointly. First we can assume by translation and rescaling that $x_0 = 0$ and $R = 1$.

The basic idea is to use the test function $\zeta = u^\beta \eta^2$ in the weak equation

$$\int a^{ij} D_j u D_i \zeta \leq (\geq) 0, \tag{5}$$

where $\eta \in W_0^{1,2}$. For now assume that $\zeta \in W_0^{1,2}$; we will come back this later. The choice of test function yields

$$\beta \int a^{ij} u^{\beta-1} D_i u D_j u \eta^2 \leq (\geq) -2 \int a^{ij} u^\beta \eta D_j u D_i \eta.$$

Now choose $\beta > 0$ if $Lu \geq 0$ weakly in B_1 and $\beta < 0$ if $Lu \leq 0$ weakly in B_1. By ellipticity and the bound on a^{ij},

$$|\beta| \lambda \int u^{\beta-1} |Du|^2 \eta^2 \leq 2\Lambda \int u^\beta \eta |Du||D\eta|$$

and so by Cauchy's inequality,

$$\int u^{\beta-1} |Du|^2 \eta^2 \leq \frac{4\Lambda^2}{\beta^2 \lambda^2} \int u^{\beta+1} |D\eta|^2. \tag{6}$$

Now assume that $\beta \neq -1$ and let $\gamma = (\beta+1)/2$ and $w = u^\gamma$ to get

$$\int |Dw|^2 \eta^2 \leq \frac{4\Lambda^2 \gamma^2}{\beta^2 \lambda^2} \int w^2 |D\eta|^2,$$

which implies

$$\int |D(w\eta)|^2 \leq C \int w^2 |D\eta|^2$$

for some constant $C = C(\Lambda/\lambda) \in (0, \infty)$ provided β remains bounded away from zero. Applying the Sobolev inequality on the left-hand-side,

$$\|w\eta\|_{L^{2\kappa}(B_1)} \leq C \|wD\eta\|_{L^2(B_1)},$$

where $\kappa = n/(n-2) > 1$ if $n \geq 3$ and $1 < \kappa < \infty$ and $C = C(n, \Lambda/\lambda) \in (0, \infty)$ is a constant. (Note that in the general case where b^i, c^i, d, f^i, g are not all zero, we also use the Hölder inequality here to deal with these terms.) Using the fact that $w = u^\gamma$ and choosing η be a cutoff function such that

$$\|u\|_{L^{2\kappa\gamma}(B_r)} \leq (C(s-r)^{-1}\gamma)^{1/\gamma}\|u\|_{L^{2\gamma}(B_s)} \tag{7}$$

if $\gamma > 0$ and $0 < r < s$ and

$$\|u\|_{L^{2\gamma}(B_s)} \leq (C(s-r)^{-1}|\gamma|)^{1/|\gamma|}\|u\|_{L^{2\kappa\gamma}(B_r)} \tag{8}$$

if $\gamma < 0$ and $0 < r < s$. Note that we let $\|u\|_{L^{p'}(B)} = \left(\int_B u^{p'}\right)^{1/p'}$ for any ball B even when $p' < 1$, though this is clearly just notation and does not represent a norm. The basic idea now is to iterate the inequalities (7) and (8).

Now the above argument is not quite rigorous yet since we need $\zeta \in W_0^{1,2}(B_1)$. Consider more generally $\zeta = G(u)\eta^2$, where $G : \mathbb{R} \to \mathbb{R}$ is C^1. To show that $\zeta \in W^{1,2}$ we typically use the fact that G' is bounded. However, when $G(t) = t^\beta$, $G'(t) = \beta t^{\beta-1}$, which becomes unbounded as $t \to \infty$ for $\beta > 1$ and becomes unbounded as $t \downarrow 0$ for $\beta < 1$. By replacing u with $u + \varepsilon$ for $\varepsilon \downarrow 0$, we can prove Theorems 1 and 2 in the special case that u is strictly positive. Moreover, when $\beta > 1$ we instead let $\zeta = G(u)\eta^2$ in (5), where

$$G(t) = \begin{cases} t^\beta & \text{if } t \leq N, \\ N^\beta + \beta N^{\beta-1}(t-N) & \text{if } t \geq N. \end{cases}$$

This yields

$$\int a^{ij}G'(u)D_iuD_ju\eta^2 \leq -2\int a^{ij}G(u)\eta D_juD_i\eta.$$

By ellipticity and the bound on a^{ij} and the fact that $G(u) \leq G'(u)u$ due to G being convex,

$$\lambda\int G'(u)|Du|^2\eta^2 \leq 2\Lambda\int G'(u)u\eta|Du||D\eta|$$

and so by Cauchy's inequality,

$$\int G'(u)|Du|^2\eta^2 \leq \frac{4\Lambda^2}{\lambda^2}\int G'(u)u^2|D\eta|^2.$$

Now if $u \in L^{2\gamma}(B_1)$, then we can let $N \to \infty$ to obtain (6) and then argue as above to obtain (7).

Now let us suppose that $Lu \geq 0$ weakly in B_1 and return to (7). Choose $\gamma = \kappa^{j-1}p/2$, $r = 1/2 + 2^{-j-1}$, and $s = 1/2 + 2^{-j}$ in (7) for $j = 1, 2, 3, \ldots, m$ to get

$$\|u\|_{L^{\kappa^m p}(B_{1/2})} \leq \prod_{j=1}^{m}(Cp)^{2\kappa^{-j+1}/p}2^{2j\kappa^{-j+1}/p}\kappa^{2(j-1)\kappa^{-j+1}/p}\|u\|_{L^p(B_1)}$$

for all $m = 1, 2, 3, \ldots$ and some constant $C = C(n, \Lambda/\lambda) \in (0, \infty)$ provided $u \in L^p(B_1)$. Using the fact that $\sum_{j=1}^\infty \kappa^{-j} < \infty$ and $\sum_{j=1}^\infty j\kappa^{-j} < \infty$ since $\kappa > 1$,

$$\|u\|_{L^{\kappa^m p}(B_{1/2})} \leq C\|u\|_{L^p(B_1)} \tag{9}$$

for all $m = 1, 2, 3, \ldots$ and some constant $C = C(n, p, \Lambda/\lambda) \in (0, \infty)$ independent of m provided $u \in L^p(B_1)$. By choosing $p = 2$, we get that $u \in L^P(B_{1/2})$ for all $P \geq 1$. It then follows as a basic fact about Lebesgue functions and (9) that $u \in L^\infty(B_{1/2})$ with

$$\sup_{B_{1/2}} u = \lim_{P\to\infty} \|u\|_{L^P(B_{1/2})} \leq C\|u\|_{L^p(B_1)}.$$

The supersolution case where $Lu \leq 0$ weakly in B_2 is a bit more complicated (it is convenient for the numbering if we set $R = 2$, instead of 1). We can again iterate (7) with $\gamma = \kappa^{-j}p/2$, $r = 3/2 - 2^{-j}$, and $s = 3/2 - 2^{-j-1}$ in (7) for $j = 1, 2, \ldots, m$ to get

$$\|u\|_{L^p(B_1)} \leq C\|u\|_{L^{\kappa^{-m}p}(B_{3/2})}$$

for all $0 < p < \kappa$, all $m = 1, 2, 3, \ldots$, and some constant $C = C(n, p, \Lambda/\lambda) \in (0, \infty)$ independent of m. Note that since for the supersolution case $\beta < 0$ and thus $\gamma < 1/2$, we need $p < \kappa$. By the Hölder inequality,

$$\|u\|_{L^p(B_1)} \leq C\|u\|_{L^{p_0}(B_{3/2})} \tag{10}$$

for $0 < p_0 < p < \kappa$, and some constant $C = C(n, p, \Lambda/\lambda) \in (0, \infty)$. We can also iterate (8) as above with $\gamma = -\kappa^{j-1}p_0/2$ for $p_0 > 0$, $r = 1/2 + 2^{-j}$, and $s = 1/2 + 2^{-j+1}$ in (7) for $j = 1, 2, \ldots, m$ to get

$$\|u\|_{L^{-p_0}(B_{3/2})} \leq C\|u\|_{L^{-\kappa^m p_0}(B_{1/2})} \tag{11}$$

for all $p_0 > 0$, all $m = 1, 2, 3, \ldots$, and some constant $C = C(n, p_0\Lambda/\lambda) \in (0, \infty)$ independent of m. Observe that (11) implies that $1/u \in L^P(B_{1/2})$ for all $p \geq 1$ and

$$\lim_{P\to\infty} \|u\|_{L^{-P}(B_{1/2})} = \lim_{P\to\infty} \frac{1}{\|1/u\|_{L^P(B_{1/2})}} = \frac{1}{\sup_{B_{1/2}}(1/u)} = \inf_{B_{1/2}} u,$$

so by letting $m \to \infty$ in (11),

$$\|u\|_{L^{-p_0}(B_{3/2})} \leq C \inf_{B_{1/2}} u. \tag{12}$$

Thus to complete the proof of Theorem 2, it suffices to show that for some $p_0 \in (0, p)$,

$$\|u\|_{L^{p_0}(B_{3/2}(0))} \leq C\|u\|_{L^{-p_0}(B_{3/2}(0))} \tag{13}$$

for some constant $C = C(n, p_0, \Lambda/\lambda) \in (0, \infty)$, as then the conclusion of Theorem 2 will follow from combining (10), (12), and (13).

3 Part 2: Proving (13)

Recall (6),

$$\int u^{\beta-1}|Du|^2\eta^2 \leq \frac{4\Lambda^2}{\beta^2\lambda^2} \int u^{\beta+1}|D\eta|^2.$$

Let $\beta = -1$ and $w = \log u$ and to get that

$$\int_{B_2} \eta^2|Dw|^2 \leq C\int_{B_2} |D\eta|^2 \tag{14}$$

for all $\eta \in W^{1,2}_0(B_2)$ for some constant $C = C(\lambda, \Lambda) \in (0, \infty)$.

By choosing η in (14) to be the cutoff function such that $0 \leq \eta \leq 1$, $\eta = 1$ on $B_{7/4}(0)$, $\eta = 0$ on $\mathbb{R}^n \setminus B_2(0)$, and $|D\eta| \leq 6$,

$$\int_{B_{7/4}(0)} |Dw|^2 \leq C \tag{15}$$

for some constant $C \in (0, \infty)$.

Choose $\eta = \phi^{\alpha Q - \beta} |w - \ell|^{Q-1}$ in (14) for a cutoff function ϕ such that $0 \leq \phi \leq 1$, $\phi = 1$ in $B_{3/2}(0)$, $\phi = 0$ on $\mathbb{R}^n \setminus B_{7/4}(0)$, and $|D\phi| \leq 6$ and constants $Q \geq 1$, ℓ, α, and β to be specified later. Since

$$\left| \frac{d}{dt} \left(|\log(t) - \ell|^Q \right) \right| = \frac{Q|\log(t) - \ell|^{Q-1}}{t}$$

remains bounded as $t \to \infty$, $|w - \ell|^{Q-1} = |\log(u) - \ell|^{Q-1} \in W^{1,2}(B_2)$. With this choice of η, (14) yields

$$\begin{aligned} \int_{B_{7/4}(0)} \phi^{2\alpha Q - 2\beta} |w - \ell|^{2Q-2} |Dw|^2 \leq & CQ^2 \int_{B_{7/4}(0)} \phi^{2\alpha Q - 2\beta - 2} |D\phi|^2 |w - \ell|^{2Q-2} \\ & + CQ^2 \int_{B_{7/4}(0)} \phi^{2\alpha Q - 2\beta} |w - \ell|^{2Q-4} |Dw|^2 \end{aligned} \tag{16}$$

for $C = C(n, \Lambda/\lambda, \nu, \alpha, \beta) \in (0, \infty)$. How we will do the usual computation, where we move the terms with derivatives of w to the left-hand-side and then use the Sobolev inequality (and Hölder inequality) to get an estimate that we will iterate. To move the derivatives of w onto the left hand side, we usually would use Cauchy's inequality. Instead we use Young's inequality to get

$$CQ^2 |w - \ell|^{2Q-4} \leq \frac{1}{2} |w - \ell|^{2Q-2} + \frac{1}{2} (C')^Q Q^{2Q-2}$$

for some constant $C' = C'(n, \Lambda/\lambda, \nu, \alpha, \beta) \in (0, \infty)$ and then substitute into (16) to get

$$\begin{aligned} \int_{B_{7/4}(0)} \phi^{2\alpha Q - 2\beta} |w - \ell|^{2Q-2} |Dw|^2 \leq & CQ^2 \int_{B_{7/4}(0)} \phi^{2\alpha Q - 2\beta - 2} |D\phi|^2 |w - \ell|^{2Q-2} \\ & + (C')^Q Q^{2Q-2} \int_{B_{7/4}(0)} \phi^{2\alpha Q - 2\beta} |Dw|^2. \end{aligned}$$

Then by (15) (below, we will ensure that $\alpha Q \geq \beta + 1$),

$$\int_{B_{7/4}(0)} \phi^{2\alpha Q - 2\beta} |w - \ell|^{2Q-2} |Dw|^2 \leq CQ^2 \int_{B_{7/4}(0)} \phi^{2\alpha Q - 2\beta - 2} |D\phi|^2 |w - \ell|^{2Q-2} + C^Q Q^{2Q-2}$$

for some constant $C = C(n, \Lambda/\lambda, \nu, \alpha, \beta) \in (0, \infty)$. Next, using $|w - \ell|^{2Q-2} \leq 1 + |w - \ell|^{2Q}$,

$$\int_{B_{7/4}(0)} |D(\phi^{\alpha Q - \beta} |w - \ell|^Q)|^2 \leq C^Q Q^{2Q} + CQ^4 \int_{B_{7/4}(0)} \phi^{2\alpha Q - 2\beta - 2} |D\phi|^2 |w - \ell|^{2Q},$$

so by the Sobolev inequality and definition of ϕ,

$$\left(\int_{B_{7/4}(0)} \phi^{2\alpha\kappa Q - 2\beta\kappa} |w - \ell|^{2\kappa Q} \right)^{1/\kappa} \leq C^Q Q^{2Q} + CQ^4 \int_{B_{7/4}(0)} \phi^{2\alpha Q - 2\beta - 2} |w - \ell|^{2Q} \tag{17}$$

for some constant $C = C(n, \Lambda/\lambda, \nu, \alpha, \beta) \in (0, \infty)$, where as above $\kappa = n/(n-2)$ if $n \geq 3$ and $\kappa > 1$ if $n = 2$. (Note that in the case where b^i, c^i, d, f^i, g are not all zero, we have to apply the Hölder inequality here and get in fact $|w - \ell|^{2Q\tau}$ in the integral on the right-hand-side, where $\tau = q/(q-2)$. Thus we have to use the interpolation inequality (20) below to get (17).) Choose β so that $\beta\kappa = \beta + 1$, i.e. $\beta = 1/(\kappa - 1)$, and thus

$$\left(\int_{B_{7/4}(0)} (\phi^\alpha |w - \ell|)^{2\kappa Q} \cdot \phi^{-2\beta-2} dX \right)^{1/\kappa} \leq C^Q Q^{2Q} + C Q^4 \int_{B_{7/4}(0)} (\phi^\alpha |w - \ell|)^{2Q} \cdot \phi^{-2\beta-2} dX.$$

Taking the $1/Q$ power of both sides and using $(a+b)^{1/Q} \leq a^{1/Q} + b^{1/Q}$ for $a, b \geq 0$ yields

$$\left(\int_{B_{7/4}(0)} (\phi^\alpha |w - \ell|)^{2\kappa Q} \cdot \phi^{-2\beta-2} dX \right)^{1/\kappa Q} \leq C Q^2 + C^{1/Q} \left(\int_{B_{7/4}(0)} (\phi^\alpha |w - \ell|)^{2Q} \cdot \phi^{-2\beta-2} dX \right)^{1/Q}. \tag{18}$$

Iterating (18) by taking $Q = \kappa^{m-1}$ for integers $m \geq m_0 + 1$, where m_0 is the least integer such that $\kappa^{m_0} \geq 2$, we obtain

$$\begin{aligned}
&\left(\int_{B_{7/4}(0)} (\phi^\alpha |w - \ell|)^{2\kappa^m} \cdot \phi^{-2\beta-2} dX \right)^{1/\kappa^m} \leq C\kappa^{2m-2} + C^{\kappa^{-m+1}} \Bigg(C\kappa^{2m-4} + C^{\kappa^{-m+2}} \Bigg(\cdots \\
&\qquad \left(C\kappa^{2m_0} + C^{\kappa^{-m_0}} \left(\int_{B_{7/4}(0)} (\phi^\alpha |w - \ell|)^{2\kappa^{m_0}} \cdot \phi^{-2\beta-2} dX \right)^{1/\kappa^{m_0}} \right) \cdots \Bigg) \Bigg) \\
&= C\kappa^{2m} + \sum_{j=m_0+1}^{m-2} C^{\sum_{l=j+1}^{m-1} \kappa^{-2l}} \cdot C\kappa^{2j} + C^{\sum_{l=m_0+2}^{m-1} \kappa^{-2l}} \cdot C \left(\int_{B_{7/4}(0)} (\phi^\alpha |w - \ell|)^{2\kappa^{m_0}} \cdot \phi^{-2\beta-2} dX \right)^{1/\kappa^{m_0}} \\
&\leq C\kappa^{2m} + C \left(\int_{B_{7/4}(0)} (\phi^\alpha |w - \ell|)^{2\kappa^{m_0}} \cdot \phi^{-2\beta-2} dX \right)^{1/\kappa^{m_0}} \qquad (19)
\end{aligned}$$

for every integer $m \geq 1$ for constants $C = C(n, \Lambda/\lambda, \nu, q, \alpha) \in (0, \infty)$ independent of m, provided $m \geq m_0 + 3$ and with obvious modification if $m = m_0 + 1, m_0 + 2$.

Now recall that by the Hölder inequality and Young's inequality, we have the following interpolation result: for any function $f \in L^{p_1}$ on an abstract measure space, $1 \leq p_0 < p_1 < \infty$, and p_t defined by $1/p_t = (1-t)/p_0 + t/p_1$ for each $t \in (0, 1)$,

$$\|f\|_{L^{p_t}} \leq \|f^{1-t}\|_{L^{p_0/(1-t)}} \|f^t\|_{L^{p_0/t}} = \|f\|_{L^{p_0}}^{1-t} \|f\|_{L^{p_1}}^t \leq (1-t)\|f\|_{L^{p_0}} + t\|f\|_{L^{p_1}}. \tag{20}$$

By letting $f = \phi^\alpha |w - \ell|$ on the measure space $B_{7/4}(0)$ with measure $\phi^{\kappa/(\kappa-1)} dX$ and choosing p_0, p_1, and t so that t is small and $p_0 = 2$, $p_1 = \kappa^m$, and $p_t = \kappa^{m_0 - 1}$ for some integer $m > m_0$,

$$\begin{aligned}
&\left(\int_{B_{7/4}(0)} (\phi^\alpha |w - \ell|)^{2\kappa^{m_0}} \cdot \phi^{-2\beta-2} dX \right)^{1/\kappa^{m_0}} \\
&\leq \left(\int_{B_{7/4}(0)} (\phi^\alpha |w - \ell|)^2 \cdot \phi^{-2\beta-2} dX \right)^{1/2} + t \left(\int_{B_{7/4}(0)} (\phi^\alpha |w - \ell|)^{2\kappa^m} \cdot \phi^{-2\beta-2} dX \right)^{1/2\kappa^m} \\
&\leq \left(\int_{B_{7/4}(0)} (\phi^\alpha |w - \ell|)^2 \cdot \phi^{-2\beta-2} dX \right)^{1/2} + tC\kappa^{2m} + tC \left(\int_{B_{7/4}(0)} (\phi^\alpha |w - \ell|)^{2\kappa^{m_0}} \cdot \phi^{-2\beta-2} dX \right)^{1/\kappa^{m_0}}
\end{aligned}$$

and so taking t small enough that $tC < 1/2$ and fixing m,

$$\left(\int_{B_{7/4}(0)} (\phi^\alpha |w - \ell|)^{2\kappa^{m_0}} \cdot \phi^{-2\beta-2} dX \right)^{1/\kappa^{m_0}} \leq C + C \left(\int_{B_{7/4}(0)} (\phi^\alpha |w - \ell|)^2 \cdot \phi^{-2\beta-2} dX \right)^{1/2}.$$

for some constant $C = C(n, \Lambda/\lambda, \nu, q, \alpha) \in (0, \infty)$. By choosing $\alpha = \beta + 1$,

$$\left(\int_{B_{7/4}(0)} (\phi^\alpha |w - \ell|)^{2\kappa^{m_0}} \cdot \phi^{-2\beta-2} dX \right)^{1/\kappa^{m_0}} \leq C + C \left(\int_{B_{7/4}(0)} |w - \ell|^2 dX \right)^{1/2}.$$

By the Poincaré inequality and (15), for some constant $\ell \in \mathbb{R}$,

$$\left(\int_{B_{7/4}(0)} (\phi^\alpha |w - \ell|)^{2\kappa^{m_0}} \phi^{-2\beta-2} dX \right)^{1/\kappa^{m_0}} \leq C + C \left(\int_{B_{7/4}(0)} |Dw|^2 \right)^{1/2} \leq C \tag{21}$$

for some constants $C = C(n, \Lambda/\lambda, \nu, q) \in (0, \infty)$. Combining (19) and (21),

$$\left(\int_{B_{7/4}(0)} (\phi^\alpha |w - \ell|)^{2\kappa^m} \cdot \phi^{-2\beta-2} dX \right)^{1/\kappa^m} \leq C\kappa^{2m}$$

By the definition of ϕ,

$$\left(\int_{B_{3/2}(0)} |w - \ell|^{2\kappa^m} dX \right)^{1/\kappa^m} \leq C\kappa^{2m} \tag{22}$$

for some constant $C = C(n, \Lambda/\lambda, \nu, q) \in (0, \infty)$.

Given any integer $j \geq 2$, choose $m \geq 1$ such that such that $2\kappa^{m-1} \leq j < 2\kappa^m$ in (22) to obtain

$$\int_{B_{3/2}(0)} |w - \ell|^j \leq (Cj)^j. \tag{23}$$

for some constant $C = C(n, \Lambda/\lambda, \nu, q) \in (0, \infty)$. Note that (23) holds true for $j = 1$.

Recall the Taylor series $e^x = \sum_{j=0}^\infty x^j/j!$ for $x \in \mathbb{R}$ and observe that this implies that

$$e^{x/e} \leq 1 + \sum_{j=1}^\infty \frac{x^j}{j^j}$$

for all $x \geq 0$. Thus

$$\int_{B_{3/2}(0)} e^{p_0|w-\ell|} \leq 1 + \int_{B_{3/2}(0)} \sum_{j=1}^\infty \frac{|w - \ell|^j}{(2Cj)^j} \leq 1 + \sum_{j=1}^\infty \frac{1}{2^j} = \frac{3}{2}.$$

where $p_0 = 1/(2eC)$. Consequently

$$\int_{B_{3/2}(0)} e^{p_0(w-\ell)} \leq \frac{3}{2} \quad \text{and} \quad \int_{B_{3/2}(0)} e^{-p_0(w-\ell)} \leq \frac{3}{2},$$

which multiplying each side yields

$$\left(\int_{B_3(0)} e^{-p_0 w}\right)\left(\int_{B_{3/2}(0)} e^{+p_0 w}\right) \leq \frac{9}{4}.$$

Since $w = \log u$,

$$\left(\int_{B_{3/2}(0)} u^{-p_0}\right)\left(\int_{B_{3/2}(0)} u^{+p_0}\right) \leq \frac{9}{4},$$

which is equivalent to (13).

References

[GT] David Gilbarg, Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer, 1998.